Determinacy in a Canonical New Keynesian Model under a Money Growth Rule

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Abstract

This note derives the analytical conditions for determinacy in a canonical New Keynesian model under an exogenous money growth rule. Under cash-when-I'm-done timing the model is always determinate while indeterminacy can arise with cash-in-advance timing.

Keywords: Determinacy; money growth rule; New Keynesian

JEL classification: E32, E40, E52, E58.

1. Introduction

A large body of the monetary policy literature has emphasized the importance of looking at monetary policy rules that lead to determinate equilibria. For instance, Clarida et. al. (2000) argue that a violation of the Taylor principle by the pre-Volcker rule may have contained the seeds of macroeconomic instability amid the Great Inflation.

Determinacy of the canonical New Keynesian model under a variety of interest rate rules has been analyzed by Bullard and Mirtra (2002) among others. Under such monetary regimes, there is no need to pay attention to the demand for money. Money demand is only usefull for purposes of implementation, backing out the ammount of money needed to achieve the target interest rate.

On the other hand, in the presence of exogenous money growth rules, money demand plays a key role for determinacy. Carlstrom and Fuerst (2003) provide the analytical conditions for determinacy in a standard production (flexible prices) and an endowment economy. To date, there is no equivalent analytical proof for determinacy in a canonical New Keynesian model, although Galí (2015)[1] states that through numerical analysis, under *cash-when-I'm-done* (CWID) timing, an exogenous money growth rule is shown to always yield determinacy. The contribution of this letter is to provide such analytical characterization. We prove that a

[1]See Galí (2015) page 75.

money growth rule always guarantees determinacy under CWID timing and we characterize the parameter range leading to indeterminacy under *cash-in-advance* (CIA) timing.

2. The Model

For the exposition of the canonical New Keynesian model, we closely follow Galí (2015), Chapter 3.

Households

The economy is populated by a continuum of mass 1 of identical and infinitely lived households. Households preferences are represented by a CRRA separable utility function:

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} + \frac{(A_t/P_t)^{1-\nu} - 1}{1-\nu} - \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

where C_t is consumption at t, A_t the nominal value of a generic asset, P_t the price level and N_t employment or hours worked. Households face the following budget constraint:

$$P_t C_t + Q_t B_t + M_t \leq B_{t-1} + M_{t-1} + W_t N_t + D_t + T_t$$

where M_t denotes money holdings in period t, B_t represents the quantity of one-period nominal riskless bonds held at t and Q_t their price, W_t denotes the nominal wage at t, T_t and D_t transfers and dividends at t.

For our choice of timing we follow Carlstrom and Fuerst (2003). $A_t = M_{t-1}$ for CIA timing and $A_t = M_t$ for CWID timing. Solving the household's maximization problem leads to the following log-linear money demand equations:

$$\textit{CIA money demand}: m_t - p_t = \frac{\sigma}{\nu} y_t - \eta i_t - \left(\frac{1-\nu}{\nu} \right) E_t\{\pi_{t+1}\} \quad (1)$$

$$\textit{CWID money demand}: m_t - p_t = \frac{\sigma}{\nu} y_t - \eta i_t \quad (2)$$

where $\eta \equiv \frac{1}{\nu(\exp\{i\}-1)}$ is the implied interest semi-elasticity of money demand and i is the steady state nominal interest rate. From now onwards we set $\nu = \sigma$, as it implies the natural assumption of unit elasticity with respect to income, required for a stable long-run money demand.

The *money-in-the-utility-function* approach, whereby real money balances are an argument of the utility function, is taken up as it covers both transaction cost and shopping time models, Feenstra (1986). In addition, since we assumed separability, neither $U_{c,t}$ nor $U_{n,t}$ depend on real money balances. This would allow oneself to derive the model without money in the utility function and then postulate an *ad-hoc* money demand as is usually done. Separability is a common assumption due to non-separability being quantitatively neglegible, see e.g. McCallum (2001) and Woodford (2011).

The household's intertemporal optimality condition, jointly with the goods market clearing condition ($y_t = c_t$), delivers the familiar log-linearized dynamic IS equation.

$$\textit{Dynamic IS equation}: \tilde{y}_t = -\frac{1}{\sigma}(i_t - E_t\{\pi_{t+1}\} - r_t^n) + E_t\{\tilde{y}_{t+1}\} \quad (3)$$

where $E_t\{\cdot\}$ relates to the expectational operator conditional on information at time t, and σ to the intertemporal elasticity of substitution and risk aversion.

The Dynamic IS equation determines the output gap,[2] $\tilde{y}_t$ for any given natural interest rate r_t^n and real interest rate: $r_t \equiv i_t - E_t\{\pi_{t+1}\}$; path.

Firms

In the canonical New Keynesian model, characterized by imperfect competition and Calvo pricing in the goods market, the following relation determines inflation, π_t, for any given output gap path.

$$\textit{New Keynesian Phillips curve}: \pi_t = \beta E_t\{\pi_{t+1}\} + \kappa\tilde{y}_t \quad (4)$$

with κ relating to the degree of price stickiness.

Monetary policy

Note that real money balances, $l_t \equiv m_t - p_t$, inflation and money growth are related through the following identity:

$$\widehat{l}_{t-1} = \widehat{l}_t + \pi_t - \Delta m_t \quad (5)$$

where Δm_t is the exogenous growth rate of the money supply

Equilibrium

The previous money demand equations, rewritten in terms of real money balances, $l_t \equiv m_t - p_t$ and the output gap, by adding and subtracting y_t^n, are used to eliminate the nominal interest rate from equation 3.

[2]The output gap is defined as the deviation of output from the natural, frictionless, output: $\tilde{y}_t \equiv y_t - y_t^n$

The described dynamic system, characterized by equations 1 or 2, 3, 4 and 5, can be summarized in the following compact form, as in Galí (2015) page 75:

$$\mathbf{A_{M,0}} \begin{bmatrix} \tilde{y}_t \\ \pi_t \\ \widehat{l}_{t-1} \end{bmatrix} = \mathbf{A_{M,1}} \begin{bmatrix} E_t\{\tilde{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \\ \widehat{l}_t \end{bmatrix} + \mathbf{B_M} \begin{bmatrix} \hat{r}_t^n \\ \hat{y}_t^n \\ \Delta m_t \end{bmatrix}$$

where:

$$\mathbf{A_{M,0}} = \begin{bmatrix} 1+\sigma\eta & 0 & 0 \\ -\kappa & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}; \mathbf{A_{M,1}} = \begin{bmatrix} \sigma\eta & \Omega & 1 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}; \mathbf{B_M} = \begin{bmatrix} \eta & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

with $\Omega \equiv \eta$ under CWID timing and $\Omega \equiv \frac{1-\sigma}{\sigma} + \eta$ for CIA timing. Some constants, that leave the dynamics of the model unaffected, are ommited and subtracted to obtain $\widehat{l}$, $\widehat{y^n}$ and $\widehat{r^n}$; where " ^" on top of a variable denotes it's deviation from the steady state.

3. Determinacy Conditions

For the previously described linear dynamic system, a stationary solution will exist and be unique if and only if $\mathbf{A_M} \equiv \mathbf{A_{M,0}}^{-1}\mathbf{A_{M,1}}$ has two eigenvalues inside the unit circle and one eigenvalue outside (or on) the unit circle, see Blanchard and Kahn (1980).

After some algebra:

$$\mathbf{A_M} = \begin{bmatrix} \frac{\sigma\eta}{1+\sigma\eta} & \frac{\eta}{1+\sigma\eta} & \frac{1}{1+\sigma\eta} \\ \frac{\kappa\sigma\eta}{1+\sigma\eta} & \frac{\kappa\eta}{1+\sigma\eta} + \beta & \frac{\kappa}{1+\sigma\eta} \\ \frac{\kappa\sigma\eta}{1+\sigma\eta} & \frac{\kappa\eta}{1+\sigma\eta} + \beta & \frac{\kappa}{1+\sigma\eta} + 1 \end{bmatrix}$$

whose eigenvalues are given by the roots of the following polynomial[3]:

$$p(x) = \underbrace{-}_{a} x^3 + \underbrace{\left(\frac{\sigma\eta + \kappa(1+\Omega)}{1+\sigma\eta} + 1 + \beta\right)}_{b} x^2 \underbrace{- \left((1+\beta)\frac{\sigma\eta}{1+\sigma\eta} + \frac{\kappa\Omega}{1+\sigma\eta} + \beta\right)}_{c} x + \underbrace{\frac{\beta\sigma\eta}{1+\sigma\eta}}_{d}$$

where $\sigma > 0$; $\eta > 0$; $\kappa > 0$ and $\beta \in (0,1)$.

Proposition

The system is determinate if and only if $\Omega > -\frac{1}{2} - \frac{(1+\sigma\eta)(1+\beta)}{\kappa} - \frac{(1+\beta)\sigma\eta}{\kappa}$

[3] A condition for the inversion of $\mathbf{A_{M,0}}$ is: $1 + \sigma\eta \neq 0$; but this condition will never be violated due to the assumptions on σ and η.

Proof

We will first explore the case where all roots are real.

Since $p(1) = \kappa/(1+\sigma\eta) > 0$ and $p(+\infty) < 0$; by continuity and Bolzano's theorem there must be at least one real root larger than 1 (λ_1) and therefore outside the unit circle. This root is also the only real root larger or equal to 1, see the Appendix.

Furthermore, we need that $p(x) > 0 \ \ \forall \, x \in (-\infty, -1]$ so the other two roots (λ_2 and λ_3) are in the interval (-1,1) and thus inside the unit circle. The previous condition will be meet if *i)* $p(-1) > 0$; *ii)* $\frac{\partial p(x)}{\partial x}|_{x=-1} < 0$ and *iii)* $\frac{\partial^2 p(x)}{\partial x^2}|_{x\in(-\infty,-1]} > 0$.

$$\frac{\partial^2 p(x)}{\partial x^2} = -6x + 2\left(\frac{\sigma\eta + \kappa(1+\Omega)}{1+\sigma\eta} + 1 + \beta\right)$$

Due to the form of $\frac{\partial^2 p(x)}{\partial x^2}$, if $\frac{\partial^2 p(x)}{\partial x^2}|_{x=-1} > 0 \longrightarrow \frac{\partial^2 p(x)}{\partial x^2}|_{x \, \in \, (-\infty,-1]} > 0$. $\frac{\partial^2 p(x)}{\partial x^2}|_{x=-1} > 0$ is met if and only if $\Omega > -1 - \frac{\sigma\eta}{\kappa} - \frac{(1+\sigma\eta)(4+\beta)}{\kappa}$; which will always hold due to $\Omega \equiv \eta > 0$ under CWID timing and $\Omega \equiv \frac{1-\sigma}{\sigma} + \eta \in (-1, +\infty)$ under CIA timing.

For condition *ii* we need that:

$$\Omega > -\frac{2}{3} - \frac{(5/3+\beta)(1+\sigma\eta)}{\kappa} - \frac{(1+\beta/3)\sigma\eta}{\kappa} \qquad (6)$$

Also, $p(-1) > 0$ if and only if:

$$\Omega > -\frac{1}{2} - (1+\beta)\frac{\sigma\eta}{\kappa} - \frac{(1+\sigma\eta)(1+\beta)}{\kappa} \qquad (7)$$

Note that condition 7 implies condition 6.

To adress the case of two (λ_2 and λ_3) complex conjugates, we write our polynomial in a more general form:

$$p(x) = -(x-\lambda_1)(x-\lambda_2)(x-\lambda_3) = -x^3 + bx^2 + cx + d$$

Where $d = \lambda_1\lambda_2\lambda_3$ is the independent component. Since $|\lambda_2| = |\lambda_3|$ and $\lambda_2\lambda_3 = d\frac{1}{\lambda_1}$ $\rightarrow |\lambda_3|^2 = d\frac{1}{\lambda_1}$.

$$d\frac{1}{\lambda_1} = \underbrace{\beta}_{\in(0,1)} \underbrace{\frac{\sigma\eta}{\lambda_1 + \lambda_1\sigma\eta}}_{<1 \ due \ to \ \lambda_1>1} < 1$$

Therefore, one root is on or outside the unit circle (λ_1) and two (λ_2 and λ_3) inside the unit circle. The dynamic linear system has a unique and stable solution under any parameter values for the case of two complex conjugates.

To see that condition 7 implies an if and only if relation, when condition 7 is not met, $p(-1) \leq 0$, and due to $p(-\infty) > 0$, by continuity and Bolzano's theorem, there is at least one real root smaller or equal to -1 (so we are in the case where all roots are real) and thus outside (or on) the unit circle. $\square$

4. Discussion

From the previous proposition we obtain some straightforward insights. Under CWID timing, $\Omega \equiv \eta > 0$; so condition 7 is always met. As a result, in this particular case, the model is determinate under any set of parameter values, corroborating the findings of Galí's numerical analysis.

Under CIA timing, however, the system might not always be determinate. To achieve indeterminacy one would need: *i)* a high degree of risk aversion, *ii)* low interest semi-elasticity of money demand and *iii)* a sufficiently high degree of price flexibility.

Note that as we get close to the case of fully flexible prices, $\kappa \to \infty$, *iii* is no longer needed, leading to the conditions obtained in Carlstrom and Fuerst (2003). This suggests that in an economy with sticky prices, an exogenous money growth rule may deliver a determinate equilibrium even when it would not under flexible prices. As price stickiness increases, the parameter range yielding determinacy under CIA timing expands. In the limit case of fix prices, $\kappa = 0$, the system is always determinate. The economic intuition for this result is that as prices are fixed there is no inflation, thus receiving money before or after going to the goods market is irrelevant as its value remains constant.

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Appendix

Proof that λ_1 is the only real root of $p(x)$ for $x \in [1, +\infty)$:

The slope of the polynomial $p(x)$ is given by:

$$\frac{\partial p(x)}{\partial x} = -3x^2 + 2\left(\frac{\sigma\eta + \kappa(1+\Omega)}{1+\sigma\eta} + 1 + \beta\right)x - \left((1+\beta)\frac{\sigma\eta}{1+\sigma\eta} + \frac{\kappa\Omega}{1+\sigma\eta} + \beta\right) \quad (1)$$

We will now explore two cases:

- *Case 1*: $\frac{\partial p(x)}{\partial x}|_{x=1} > 0$

Since $\frac{\partial p(x)}{\partial x}|_{x=-\infty} < 0$; $\frac{\partial p(x)}{\partial x}|_{x=1} > 0$; and $\frac{\partial p(x)}{\partial x}|_{x=+\infty} < 0$; by continuity and Bolzano's theorem there is a root of equation 1 between $-\infty$ and 1 and another root between 1 and $+\infty$. Since equation 1 is a polynomial of second degree it has only two roots, which implies that there is one and only one root for equation 1 between 1 and $+\infty$. All of this implies that there is only one sign change in the slope of $p(x)$ for the interval $(1, +\infty)$. Therefore, due to $p(1) > 0$; $\frac{\partial p(x)}{\partial x}|_{x=1} > 0$ and there being only one sign change for $x \geq 1$, λ_1 is the only real root of $p(x)$ for $x \in [1, +\infty)$.

- *Case 2*: $\frac{\partial p(x)}{\partial x}|_{x=1} \leq 0$

For $\frac{\partial p(x)}{\partial x}|_{x=1} \leq 0$; it must be the case that:

$$\beta + (1-\beta)\frac{\sigma\eta}{1+\sigma\eta} + \frac{2\kappa}{1+\sigma\eta} + \frac{\kappa\Omega}{1+\sigma\eta} \leq 1 \quad (2)$$

Now, the second derivative of $p(x)$ is:

$$\frac{\partial^2 p(x)}{\partial x^2} = -6x + 2\left(\frac{\sigma\eta + \kappa(1+\Omega)}{1+\sigma\eta} + 1 + \beta\right)$$

Note that if $\frac{\partial^2 p(x)}{\partial x^2}|_{x=1} < 0 \rightarrow \frac{\partial^2 p(x)}{\partial x^2}|_{x\geq 1} < 0$. If the second derivative is negative at $x = 1$ the function will be strictly concave for all $x \geq 1$. For $\frac{\partial^2 p(x)}{\partial x^2}|_{x=1} < 0$ to be the case it must be that:

$$\frac{\sigma\eta}{1+\sigma\eta} + \frac{\kappa}{1+\sigma\eta} + \frac{\kappa\eta}{1+\sigma\eta} + \beta - 1 < 1 \quad (3)$$

If condition 2 is met, 3 must also be met and therefore $p(x)$ is strictly concave for all $x \geq 1$. All of this implies that since $p(1) > 0$ and $\frac{\partial p(x)}{\partial x}|_{x=1} \leq 0$, λ_1 is the only real root of $p(x)$ for $x \in [1, +\infty)$.